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ON AUTOMORPHISM GROUPS OF SOME FAMILIES OF GRAPHS

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Abstract

Automorphism groups of graphs are quite significant part of algebraic graph theory. It reflect the symmetry of the corresponding graph, which provides maximum complete description about graph structure.

Our present study is focused on investigating the automorphism group of various families of graphs. We also address some directly related topics following our discussion of automorphisms of graphs like, vertex transitivity, edge transitivity, arc transitivity, half arc transivity, geodesic transitivity etc. We studied the following graphs in this thesis:

- Generalized Andrásfai graphs.
- Generalized Pappus graphs.
- Generalized Holt graphs.
- A family of geodesic-transitive graphs.
- A family of quasi-strongly regular Cayley graphs.

Our study provides nice insights on the algebraic as well as graph theoretic aspects of these graphs.

Chapter 1

Introduction

The thesis deals with the "symmetry of families of graphs". The study of different families of graphs with respect to their group of symmetries is an important aspect of modern algebraic graph theory. Like groups, the symmetry of graphs is measured by its automorphism group of that graph. The contents of the proposed thesis is divided into six tentative chapters as follows:

Chapter - 1 : Introduction
Chapter - 2 : Generalized Andrásfai Graphs
Chapter - 3 : Generalized Pappus Graphs
Chapter - 4 : A Family of Tetravalent Half-arc-transitive Graphs
Chapter - 5 : On Automorphism Group of a Family of Symmetric Graphs
Chapter - 6 : On a family of quasi-strongly regular Cayley graphs

The following are the list of publications related to the proposed thesis:

1. **S. Biswas** and A. Das, "A family of tetravalent half-arc-transitive graphs," Proceedings Mathematical Sciences, vol. 131(2), 2021. `https://doi.org/10.1007/s12044-021-00625-8`

2. **S. Biswas** and A. Das, "A generalization of Pappus graph," Electronic Journal of Graph Theory and Applications, vol. 10(1), 2022. `http://dx.doi.org/10.5614/ejgta.2022.10.1.25`

3. **S. Biswas**, A. Das, and M. Saha, "Generalized Andrásfai Graph," Discussiones Mathematicae - General Algebra and Applications, vol. 42(2), pp. 449–462, 2022. `https://doi.org/10.7151/dmgaa.1401`

4. **S. Biswas**, "On Automorphism Group of a Family of Symmetric Graphs". (**Submitted**)

5. **S. Biswas** and A. Das, "On a family of quasi-strongly regular Cayley graphs". (**Submitted**)

Chapter 2

Generalized Andrásfai Graphs

Andrásfai graphs, denoted by $And(k)$, is a family of triangle-free circulant graphs, named after Bela Andrásfai [3]. It is a Cayley graph on $\mathbb{Z}_n$ where $n = 3k-1$ and the connection set $S = \{[1], [4], \ldots, [3k-2]\}$, where $[x]$ denote the equivalence class of x in $\mathbb{Z}_n$. In this chapter, we introduce a generalization of this family of circulants and study its automorphisms and structural properties. For definitons and terminologies, readers are referred to [11].

Let us introduce the new family of circulants graphs $GA(t, k)$.

Definition 2.0.1 *Let $t, k \geq 2$ be two positive integers and set $n = t(k-1)+2$. Define $GA(t, k)$ to be the Cayley graph on $\mathbb{Z}_n$ with connection set $S = \{[1], [t+1], [2t+1], \ldots, [t(k-1)+1]\}$.*

Clearly, $GA(3, k) = And(k)$ and hence $GA(t, k)$ is a generalization of the family of Andrásfai graphs. Being a circulant graph, $GA(t, k)$ is a k-regular, vertex-transitive, Hamiltonian graph.

2.1 Structural Properties of $GA(t, k)$

In this section, we study some structural properties of $GA(t, k)$, like girth, diameter, chromatic number, domination number etc.

Proposition 2.1.1 *$GA(t, k)$ is triangle-free and the girth is 4.*

Theorem 2.1.1 *$GA(t, k)$ is bipartite if and only if t is even.*

Corollary 2.1.2 *If $t = 2$, then $GA(t, k)$ is a complete bipartite graph with equal partite-sets.*

Proposition 2.1.3 *If t is odd, the odd girth of $GA(t, k)$ is $t + 2$.*

Theorem 2.1.2 *The chromatic number of $GA(t, k)$ is* $= \begin{cases} 2, & \text{if } t \text{ is even,} \\ 3, & \text{otherwise.} \end{cases}$

Theorem 2.1.3 *The diameter of* $GA(t,k)$ *is* $\left\lceil \frac{t+1}{2} \right\rceil$.

Theorem 2.1.4 *The domination number of* $GA(t,k)$ *is less than or equal to* t. *If* $k > 2t+1$, *then it is equal to* t.

2.2 Automorphism Group of $GA(t,k)$

We denote the automorphsim group of $GA(t,k)$ by $A(t,k)$. If $t=2$, i.e., $n=2k$, then $GA(t,k)$ is a complete bipartite graph with equal partite sets each of size k and hence $A(2,k) \cong S_k \times S_k \times \mathbb{Z}_2$. Also, if $k=2$, then $GA(t,k) \cong C_n$ is a cycle of length n and hence $A(t,2) \cong D_n$. Thus for the rest of the section, we assume $t,k \geq 3$. Since $GA(t,k)$ is a Cayley graph, $A(t,k)$ contains a regular subgroup isomorphic to $\mathbb{Z}_n$. Note that $\varphi_a : GA(t,k) \to GA(t,k)$ defined by $\varphi_a([x]) = [x+a]$ is an automorphism of $GA(t,k)$, and $\{\varphi_a : 0 \leq a \leq n-1\}$ forms a regular cyclic subgroup of $A(t,k)$ generated by φ_1 which is isomorphic to $\mathbb{Z}_n$. Also note that $\tau : GA(t,k) \to GA(t,k)$ given by $\tau([x]) = [n-x]$ forms an automorphism of $GA(t,k)$ and $\tau\varphi_1\tau = \varphi_1^{-1}$. Thus $D_n \cong H = \langle \varphi_1, \tau \rangle$ forms a subgroup of $A(t,k)$. However, we claim that for $t \neq 2$, $A(t,k) = H$.

Theorem 2.2.1 *For* $t \neq 2$, $A(t,k) = \langle \varphi_1, \tau \rangle \cong D_n$.

Corollary 2.2.1 *For* $t \neq 2$, $GA(t,k)$ *is a normal Cayley graph.*
Proof : As $GA(t,k) = Cay(\mathbb{Z}_n, S)$ and $A(t,k) \cong D_n$, the regular subgroup of $A(t,k)$ is of index 2 and hence, normal in $A(t,k)$. □

Corollary 2.2.2 $GA(t,k)$ *is edge-transitive if and only if* $t=2$ *or* $k=2$. *If* $t,k>2$, *it has* $\lceil \frac{k}{2} \rceil$ *many edge-orbits.*

Corollary 2.2.3 *For* $t \neq 2$, *the determining number of* $GA(t,k)$ *is* 2.

2.3 Open Issues

In this chapter, we introduced a new family of circulants, called Generalized Andrásfai graphs, and studied its various parameters like diameter, girth, domination number etc. The full automorphism group and determining number is also computed.

As circulant graphs are an important class of interconnection networks in parallel and distributed computing, two important questions pertaining to generalized Andrásfai graphs for further research are to determining its metric dimension and spectrum. It is worth mentioning that recently in [17], authors computed the metric dimension of Andrásfai graphs.

Chapter 3

Generalized Pappus Graphs

Pappus graph is a bipartite cubic graph with 18 vertices and 27 edges, formed as the Levi graph of the Pappus configuration. It is named after Pappus of Alexandria who is believed to have discovered the "hexagon theorem" describing the Pappus configuration. Recently, [19] proposed a group theoretic generalization of Pappus configuration from a projective geometric viewpoint. Our goal is to generalize it from a graph theoretic viewpoint. To begin with, we define the Genralized Pappus graph next. For other definitons and terminologies, readers are referred to [11].

Definition 3.0.1 *Let $m \geq 3$ be a positive integer and set $n = 2m$. The generalized Pappus graph $\Gamma(m)$ is defined on the vertex set $V = \{x_i, y_i, z_i : i \in \mathbb{Z}_n\}$, where x_i's, y_i's and z_i's are called the outer vertices, middle vertices and inner vertices respectively. There are four types of edges between these vertices, namely outer edges of the form $x_i \sim x_{i+1}$, spoke edges of the form $x_i \sim y_i$, middle edges of the form $y_i \sim z_{i+1}$ and $y_i \sim z_{i-1}$ and inner edges of the form $z_i \sim z_{i+m}$.*

3.1 Automorphism Group of $\Gamma(m)$

We start by noting some automorphisms of $\Gamma(m)$. It can be easily checked that $\rho : \Gamma(m) \to \Gamma(m)$ and $\tau : \Gamma(m) \to \Gamma(m)$ defined by

$$\rho : \begin{array}{l} x_i \mapsto x_{i+1} \\ y_i \mapsto y_{i+1} \\ z_i \mapsto z_{i+1} \end{array} \qquad \tau : \begin{array}{l} x_i \mapsto x_{-i} \\ y_i \mapsto y_{-i} \\ z_i \mapsto z_{-i} \end{array}$$

are automorphisms of $\Gamma(m)$ and $\circ(\rho) = n; \circ(\tau) = 2$ and $\tau\rho\tau = \rho^{-1}$. Thus $H = \langle \rho, \tau \rangle \cong D_n$, the dihedral group of order $2n$. Moreover, if m is odd, it can be shown that $\sigma : \Gamma(m) \to \Gamma(m)$ given by

$$\sigma : \begin{array}{ll} x_i \mapsto y_{i+m} & \text{if } i \text{ is even} \\ y_i \mapsto x_{i+m} & \\ z_i \mapsto y_i & \end{array} \qquad \sigma : \begin{array}{ll} x_i \mapsto z_{i+m} & \text{if } i \text{ is odd} \\ y_i \mapsto z_i & \\ z_i \mapsto x_{i+m} & \end{array}$$

is an automorphism of $\Gamma(m)$ which does not belong to H. Also we have $\circ(\sigma) = 3$, $\sigma\rho\sigma = \rho$ and $\sigma\tau = \tau\sigma$.

Theorem 3.1.1 *If m is odd, then $\Gamma(m)$ is a Cayley graph.*

Theorem 3.1.2 *If m is even, then $Aut(\Gamma(m)) = H = \langle \rho, \tau \rangle$.*

Theorem 3.1.3 *If m is odd and $m \neq 3, 9$, then $Aut(\Gamma(m)) = \langle \rho, \tau, \sigma \rangle$.*

Theorem 3.1.4 *Let $m \geq 3$ and $\Gamma(m)$ be the Generalized Pappus graph. Then*

$$Aut(\Gamma(m)) = \begin{cases} \langle \rho, \tau \rangle & \text{if } m \text{ is even.} \\ \langle \rho, \tau, \sigma \rangle & \text{if } m \text{ is odd and } m \neq 3, 9 \\ \langle \rho, \tau, \sigma, \zeta \rangle & \text{if } m = 9 \\ \langle \rho, \tau, \sigma, \eta, \varepsilon_0 \rangle & \text{if } m = 3. \end{cases}$$

□

Corollary 3.1.1 *The following are true:*

1. *$\Gamma(3)$ and $\Gamma(9)$ are arc-transitive.*
2. *For $m \neq 3, 9$, $\Gamma(m)$ is not edge-transitive.*
3. *$\Gamma(m)$ is Cayley if and only if m is odd.*

3.2 Open Issues

Another thing which is very much related to graphs with large symmetry groups, is its hamiltonicity, viz, Lovasz' conjecture. Our observation based on first few values of m, made us to strongly believe that $\Gamma(m)$ is Hamiltonian for all value of $m \geq 3$. This can be an interseting topic for further research.

Chapter 4

A Family of Tetravalent Half-arc-transitive Graphs

The study of half-arc-transitive graphs was initiated by Tutte [20], who proved that any half-arc-transitive graph is of even degree. Since any connected 2-regular is a cycle and a cycle is arc-transitive, the first possibility of finding a half-arc-transitive graph is a 4-regular or tetravalent graph. The first examples of tetravalent half-arc-transitive graphs were given by Bouwer [4] and the smallest example was given by Holt in [14], which is the smallest example of half-arc-transitive graph, the *Holt graph* on 27 vertices. Though numerous papers have been published in the last 50 years, the classification of tetravalent half-arc-transitive graphs is not yet complete. In absence of a complete classification, two major approaches have been fruitful so far: the first is to characterize half-arc-transitive graphs of some particular orders like $p^3, p^4, p^5, pq, 2pq$ etc [6],[9],[10],[21], and the second is to come up with infinite families of half-arc-transitive graphs [5],[24].

In [2], Alspach *et.al.* constructed an infinite family of tetravalent graphs $M(a; m, n)$ and proved that

Theorem 4.0.1 ([2], Theorem 3.3) *Let $n \geq 9$ be odd and $a^3 \equiv 1 \ (mod\ n)$. Then $M(a; 3, n)$ is half-arc-transitive.*

In this chapter, we prove that $M(a; 3, n)$ is half-arc-transitive for all integers n except 7 and 14. For this, we redefine $M(a; 3, n)$ (as $\Gamma(n, a)$ in Definition 4.0.1) in a different way which helps us to prove the half-arc-transitivity of the entire family (not only when n is odd). It turns out that $\Gamma(n, a)$ is family of Cayley graphs of order $3n$ and for $(n, a) = (9, 4)$, we get the Holt graph. In fact, our definition (Definition 4.0.1) is a generalization of an alternative construction of the Holt graph (see the last paragraph of [14]).

Alspach *et.al.* constructed an infinite family of tetravalent graphs $M(a; m, n)$ and proved that if $n \geq 9$ be odd and $a^3 \equiv 1 \ (mod\ n)$, then $M(a; 3, n)$ is half-arc-transitive. In this chapter, we show that if $a^3 \equiv 1 \ (mod\ n)$, then $M(a; 3, n)$ is an infinite family of tetravalent half-arc-transitive Cayley graphs for all integers n except 7 and 14.

Definition 4.0.1 Let n be a positive integer such that $3 \mid \varphi(n)$, where φ denotes the Euler Totient function. Then $\mathbb{Z}_n^*$, the group of units of $\mathbb{Z}_n$, is a group of order a multiple of

3. Let a be an element of order 3 in $\mathbb{Z}_n^*$ and $b \equiv a^2 \ (mod\ n)$. Define $\Gamma(n,a)$ to be the graph with vertex-set $\mathbb{Z}_n \times \mathbb{Z}_3$ and the edge-set composed of edges of the form $(i,j) \sim (ai \pm 1, j-1)$ and $(i,j) \sim (bi \pm b, j+1)$, where the operations in the first and second coordinates are done modulo n and modulo 3, respectively.

It is obvious that $\Gamma(n,a)$ is tetravalent. One can check that $\Gamma(n,a)$ is suitable redefinition of $M(a;3,n)$ and $\Gamma(9,4)$ is the Holt graph. It is also to be noted that for a particular n, we can have two graphs, $\Gamma(n,a)$ and $\Gamma(n,b)$. However, these two graphs are isomorphic via the automorphism $\tau : \Gamma(n,a) \to \Gamma(n,a^2)$ defined by $\tau(i,j) = (ai,-j)$. So, without loss of generality, we assume that $a < b$, where $a, b \in \{2,\ldots,n-2\}$.

On the other hand, let n be a positive integer such that a_1, b_1, a_2, b_2 are four elements of order 3 in $\mathbb{Z}_n^*$ with $a_1 b_1 \equiv 1 \ (mod\ n)$ and $a_2 b_2 \equiv 1 \ (mod\ n)$. Then, by the above argument, $\Gamma(n,a_1) \cong \Gamma(n,b_1)$ and $\Gamma(n,a_2) \cong \Gamma(n,b_2)$. However, $\Gamma(n,a_1)$ may not be isomorphic to $\Gamma(n,a_2)$. For example, if $n = 63$, we have $4 \cdot 16 \equiv 1 \ (mod\ 63)$ and $22 \cdot 43 \equiv 1 \ (mod\ 63)$, but $\Gamma(63,4)$ is not isomorphic to $\Gamma(63,22)$, as the odd girth of $\Gamma(63,4)$ is 9, whereas that of $\Gamma(63,22)$ is 21.

The definition of $\Gamma(n,a)$ requires that $3|\varphi(n)$. We discuss the form of n for which this holds. Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where the p_i are primes. Then $\varphi(n) = {p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}\cdots{p_k}^{\alpha_k-1}(p_1-1)(p_2-1)\cdots(p_k-1)$. As $3|\varphi(n)$, either $3|{p_i}^{\alpha_i-1}$ or $3|(p_i-1)$ for some i, i.e., $9|n$ or $p_i \equiv 1 \ (mod\ 3)$ for some i. Thus n is either of the form $9t$ or pt where p is a prime of the form 1 $(mod\ 3)$ and t is a positive integer.

At this junction, it is important to note the difference in the proofs of [2] and ours:

1. First, the proof techniques are entirely different: While their proof is built on semiregular automorphisms and blocks, ours is based on 6-cycles present in the graph.

2. Second and most importantly, we prove that $\Gamma(n,a)$ is half-arc-transitive for all n except 7 and 14, i.e., n is not necessarily odd, so that we prove the result for a larger family of graphs.

4.1 Automorphisms of $\Gamma(n,a)$

Let $G = \mathsf{Aut}(\Gamma(n,a))$. We start by noting the following automorphisms of $\Gamma(n,a)$:

$$\alpha : (i,j) \mapsto (i + a^{-j}, j); \qquad \beta : (i,j) \mapsto (i, j+1); \qquad \gamma : (i,j) \mapsto (-i, j).$$

It can be shown that $\alpha, \beta, \gamma \in G$ and $\circ(\alpha) = n, \circ(\beta) = 3$ and $\circ(\gamma) = 2$. Moreover, we have the following relations: $\alpha\beta = \beta\alpha^{a^2}$, $\alpha\gamma = \gamma\alpha^{-1}$ and $\beta\gamma = \gamma\beta$.

Theorem 4.1.1 *$\Gamma(n,a)$ is a Cayley graph.*

Theorem 4.1.2 ([16], Theorem 3.3) *Every connected Cayley graph of a semidirect product of a cyclic group of prime order by an abelain group of odd order is Hamiltonian.* □

Corollary 4.1.1 *If n is odd then $\Gamma(n,a)$ is Hamiltonian .*

Theorem 4.1.3 *$\Gamma(n, a)$ is edge-transitive.*

Theorem 4.1.4 *If φ is an automorphism of $\Gamma(n, a)$ such that $n \neq 7, 14$ and $\varphi((0, 0) = (0, 0)$ and $\varphi((b, 1)) = (1, 2)$, then $\varphi((1, 2)) \notin \{(b, 1), (-b, 1), (-1, 2)\}$.* □

Thus, from Theorem 4.1.1, Theorem 4.1.3 and Theorem 4.1.4, we obtain the following result:

Theorem 4.1.5 *$\Gamma(n, a)$ is half-arc-transitive if $n \neq 7, 14$.* □

4.2 Open Issues

In this chapter, we proved the half-arc-transitivity of an infinite family of tetravalent Cayley graphs. However, a few issues are still pending and can be topics for further research.

1. **Full Automorphism Group:** It was shown that $\langle \alpha, \beta, \gamma \rangle$ is a subgroup of the full automorphism group. It remains to be shown (as observed in SageMath) that $G = \langle \alpha, \beta, \gamma \rangle$ for $n \neq 7, 14$.

2. **Structural Properties of $\Gamma(n, a)$:** We have shown that $\Gamma(n, a)$ is Hamiltonian if n is odd. However, the Hamiltonicity for even values of n is still unresolved. Similarly, other structural properties like girth, diameter, domination number are few open issues.

Chapter 5

On Automorphism Group of a Family of Symmetric Graphs

5.1 Definitions and Preliminaries

Going as per the notations used in [15], Ω_n denotes the power set of the set $[n] = \{1, 2, \ldots, n\}$, Ω'_n, Ω''_n and Ω^i_n contains all subsets of $[n]$ with odd, even and 'i' size respectively, where $i \in [n]$. Let $X \in \Omega_n$, we denote complement of X in $[n]$ by X^c, i.e., $X^c = [n] \setminus X$. $H(n,k)$ is the graph with vertex set Ω_n and any two sets X, $Y \in \Omega_n$, $X \sim Y$ if and only if $|X \triangle Y| = k$, where $\triangle$ denotes the symmetric difference of sets. Therefore $H(n,0)$ is an empty graph with no edges and $H(n,n) = 2^{n-1}K_2$, where K_2 is the complete graph with two vertices. Consider $H'(n,k)$ and $H''(n,k)$ be the subgraphs of $H(n,k)$ induced by Ω' and Ω'' respectively.

Let $\sigma \in S_n$ and $X = \{x_1, x_2, \ldots, x_m\}$. Define $\sigma(X) = \emptyset$ if $X = \emptyset$ and $\sigma(X) = \{\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_m)\}$ if $X \neq \emptyset$. Let X be an arbitrary fixed element in Ω_n. Define $\rho_X : \Omega_n \to \Omega_n$ such that $\rho_X(Y) = X \triangle Y$, for all $Y \in \Omega_n$ and $H_n = \{\rho_X : X \in \Omega_n\}$.

5.2 When $n = 2k - 1$

In this section will calculate the full automorphism group of $H(n,k)$ where $n = 2k-1$ and $k \geq 2$.

Clearly, Ω_n is an abelian group under the operation $\triangle$. Now Ω_n is isomorphic to $\mathbb{Z}_2^n$ by the map $\chi : \Omega_n \to \mathbb{Z}_2^n$ such that $\chi(X)$ is a n tuple with ith position 1 and jth position 0 if $i \in X$ and $j \notin X$, i.e., $\chi(X)$ is the *characteristic vector* of X. So we can consider Ω_n as a vector space over $\mathbb{Z}_2$, let us denote this vector space by V. Let $\sigma \in S_{n+1}$ such that $t = \sigma^{-1}(n+1)$.
Define a linear map $T_\sigma : V \to V$ such that

$$T_\sigma(v_1, v_2, \ldots, v_n) = (v_{\sigma^{-1}(1)} + v_t, v_{\sigma^{-1}(2)} + v_t, \ldots, v_{\sigma^{-1}(n)} + v_t), \text{ where } v_{n+1} = 0.$$

Lemma 5.2.1 *T_σ is a bijective linear transformation.*

Lemma 5.2.2 *If $wt(v) = k$ then $wt(T_\sigma(v)) = k$.*

Therefore from Lemmas 5.2.1 and 5.2.2 we have if $n = 2k-1$ then $|X \triangle Y| = k$ if and only if $wt(\chi(X) \oplus \chi(Y)) = k$, hence $wt(T_\sigma(\chi(X) \oplus \chi(Y))) = k$, i.e., $wt((T_\sigma \circ \chi)(X) \oplus (T_\sigma \circ \chi)(Y)) = k$. So $\chi^{-1} \circ T_\sigma \circ \chi$ is an automorphism of $H(2k-1, k)$, call it f_σ. Therefore $f_\sigma : \Omega_n \to \Omega_n$ such that

$$f_\sigma(X) = \begin{cases} \sigma(X) & \text{when } t \notin X \\ (\sigma(X \setminus \{t\}))^c & \text{when } t \in X. \end{cases} \tag{5.1}$$

where $t = \sigma^{-1}(n+1)$.

Let $\Gamma = Aut(H(2k-1, k))$ and $\Gamma_{n+1} = \{f_\sigma : \sigma \in S_{n+1}\}$.

Theorem 5.2.1 *Γ_{n+1} is isomorphic to S_{n+1}.*

Theorem 5.2.2 *$Aut(H(2k-1, k)) = \Gamma = H_n \Gamma_{n+1}$.*

5.3 When k is an odd integer and $n = 2k+1$

In this section, we will investigate the full automorphism group of $H(n, k)$ where k is an odd integer and $n = 2k+1$. Let $\Gamma' = Aut(H(2k+1, k))$ and $\Gamma'_\emptyset$ be the stabilizer of null set in Γ'. Therefore if $f \in \Gamma'_\emptyset$ then $f(\Omega_n^k) = (\Omega_n^k)$.

Let $\sigma \in S_{n+1}$ such that $\sigma(t) = n+1$ where $t \in [n]$. Define $f^\sigma : \Omega_n \to \Omega_n$ such that $f^\sigma(\{t\}) = \{\sigma(n+1)\} = \{t'\}$ and of course $f^\sigma([n]) = [n]$. Now we have $n-1$ many subsets of $S = [n] \setminus \{t'\}$ of size $n-2$ such that size of the intersection of any two subsets is $n-3$. Take $f^\sigma(\{i\}) = S \setminus \{\sigma(i)\}$ for all $i \in [n] \setminus \{t\}$ and

$$f^\sigma(X) = \begin{cases} \sigma(X) & \text{if } |X| \text{ is even and } t \notin X \\ (\sigma(X \setminus \{t\}))^c & \text{if } |X| \text{ is even and } t \in X \\ (\sigma(X \cup \{n+1\}))^c & \text{if } |X| \text{ is odd and } t \notin X \\ \sigma(X \cup \{n+1\} \setminus \{t\}) & \text{if } |X| \text{ is odd and } t \in X. \end{cases} \tag{5.2}$$

If $\sigma(n+1) = n+1$ then we take $f^\sigma(X) = \sigma(X)$ for all $X \in \Omega_n$. Let $\Gamma^{n+1} = \{f^\sigma : \sigma \in S_{n+1}\}$. As k is odd integer, hence by some basic set theoretical calculations we can show that $|X \triangle Y| = k \Leftrightarrow |f^\sigma(X) \triangle f^\sigma(Y)| = k$. Clearly $\Gamma^{n+1} \subseteq \Gamma'_\emptyset$.

By some coputations we can show that Γ^{n+1} is a subgroup of Γ'.

Theorem 5.3.1 *Γ^{n+1} is isomorphic to S_{n+1}.*

Theorem 5.3.2 *$Aut(H(2k+1, k)) = \Gamma' = H_n \Gamma^{n+1}$.*

5.4 When k is an even integer and $n = 2k+1$

Theorem 5.4.1 *$Aut(H(n, k)) = H_n S_n$, where k is an even integer and $n = 2k+1$.*

5.5 When $n \neq 2k-1, 2k, 2k+1$

We already have that H_nS_n is a subgroup of $Aut(H(n,k))$. In this section, we will prove the equality for the cases $n \neq 2k-1, 2k, 2k+1$.

Theorem 5.5.1 *$Aut(H(n,k)) = H_nS_n$, where k is an odd integer and $n \geq 2k+2$.*

Theorem 5.5.2 *$Aut(H(n,k)) = H_nS_n$, where k is an odd integer and $n \leq 2k-2$.*

Theorem 5.5.3 *$Aut(H'(n,k)) = H'_nS_n$ and $Aut(H''(n,k)) = H''_nS_n$, where k is an even integer and $n \geq 2k+2$.*

Theorem 5.5.4 *$Aut(H'(n,k)) = H'_nS_n$ and $Aut(H''(n,k)) = H''_nS_n$, where k is an even integer and $n \leq 2k-2$.*

5.6 Geodesic Transitivity

In this section we give full characterization of all n and k for which $H(n,k)$ is geodesic transitive.

Theorem 5.6.1 *$H(n,1)$ and $H(n,2)$ are geodesic transitive.*

Theorem 5.6.2 *$H(n,n-1)$ is geodesic transitive.*

Theorem 5.6.3 *$H(n,k)$ is not 2-geodesic transitive for $k \geq 3$, $n \neq k+1$ and $(n,k) \neq (5,3),\ (6,4), (7,4)$.*

5.7 Open Issues

In this chapter we have found the full automorphism group except for the case $n = 2k$, so this case is still an open issue. M. Afkhami *et.al.* [1] generalized $H(n,k)$ and defined a new family of Cayley graph, so the automorphism group of this family can be an interesting topic to investigate. Exploring the fixing number of this family of graphs, a crucial parameter contingent upon the automorphism group, could be an interesting topic for additional research.

Chapter 6

On a family of quasi-strongly regular Cayley graphs

Strongly regular graphs (SRG) is an important family of graphs generating attention of lot of researchers both due its rich theoretical implications as well as applications to various other fields. Consequently various generalizations of strong regularity, like *Deza graphs* [7],[8], *generalized strongly regular graphs* [13], *quasi strongly regular graphs* (QSRG) [12],[22],[23], has emerged over time. In this chapter, we construct a family of QSRG Cayley graphs by generalizing a well-known construction of SRG's from groups.

Definition 6.0.1 *Let G be a group of order $n \geq 5$, H be a subgroup of G and $S = \{(g,e),(e,g),(g,g) : g \in G \setminus H\}$. Define $\Gamma_H(G)$ to be the Cayley graph of $G \times G$ with respect to the connection set S, i.e., two vertices (g_1,g_2), (g_3,g_4) are adjacent if and only if $(g_1g_3^{-1}, g_2g_4^{-1}) \in S$.*

6.1 $\Gamma_H(G)$ is QSRG

Theorem 6.1.1 *Let H be a non trivial proper subgroup of G and $[G:H] = l$.*

1. *If $l = 2$ then $c-set = \{0, 2, |G| - |H|\}$.*
2. *If $l > 2$ and H is normal in G then*
   - *$|H| = 2$ implies $c-set = \{2, 6, |G| - |H|\}$.*
   - *$|H| > 2$ implies $c-set = \{0, 2, 6, |G| - |H|\}$.*
3. *If $l > 2$ and H is not normal in G then*
   - *$|H| = 2$ implies $c-set = \{2, 4, 6, |G| - |H|\}$.*
   - *$|H| > 2$ implies $c-set = \{0, 2, 4, 6, |G| - |H|\}$.*

6.2 Automorphism Group of $\Gamma_H(G)$

Let $\mathcal{G} = Aut(\Gamma_H(G))$, $\mathcal{G}_0$ be the stabilizer of the vertex (e, e) in $\mathcal{G}$.

Corollary 6.2.1 *If H is normal in G then $\mathcal{G}_0 \cong Aut_H(G) \times Sym(3)$.*

Corollary 6.2.2 *If H is not normal in G, then $\mathcal{G}_0 \cong Aut_H(G) \times \mathbb{Z}_2$.*

Theorem 6.2.1 *The automorphism group $\mathcal{G}$ of $\Gamma_H(G)$ is the Zappa-Szep product of $G \times G$ and $\mathcal{G}_0$, i.e., $\mathcal{G} = \mathcal{L}\mathcal{G}_0$ with $|\mathcal{L} \cap \mathcal{G}_0| = 1$, where $\mathcal{L} \cong G \times G$ denotes the left-translation subgroup of $\mathcal{G}$ and $\mathcal{G}_0$ is the stabilizer of the vertex (e, e) in $\mathcal{G}$.*

Corollary 6.2.3 *If G is abelian, then $\mathcal{L}$ is a normal subgroup of $\mathcal{G}$, i.e., $\mathcal{G} \cong \mathcal{L} \rtimes \mathcal{G}_0$.*

6.3 Transitivity of $\Gamma_H(G)$

As $\Gamma_H(G)$ is a Cayley graph, it is vertex-transitive. However it may not be always edge-transitive. In this section, we investigate the necessary and sufficient conditions for edge(arc)-transitivity of $\Gamma_H(G)$. In fact, we prove the following theorem.

Theorem 6.3.1 *Let G be a finite group and H be a subgroup of G. Then the following are equivalent:*

1. *$\Gamma_H(G)$ is edge-transitive.*
2. *$\Gamma_H(G)$ is arc-transitive.*
3. *H is normal in G and $Aut_H(G)$ acts transitively on $G \setminus H$.*

Corollary 6.3.1 *If $\Gamma_H(G)$ is edge-transitive, then G/H is elementary abelian group.*

6.4 Open Issues

In this chapter, we introduced a family of Cayley QSRGs and investigated its automorphism group and other transitivity properties. We conclude the chapter by listing below some natural questions which arise from this work and can be of interest to researchers.

1. The converse of Corollary 6.3.1 is not true, in general. For example, one can check using SAGEMATH [18] or otherwise that if $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ and $H \cong \mathbb{Z}_4$ be the unique cyclic subgroup of G of order 4, then though $G/H \cong \mathbb{Z}_2$, $\Gamma_H(G)$ is not edge-transitive. So it is natural to ask whether there exists some condition under which the converse holds?

2. The family of QSRGs constructed in this chapter are also Cayley graphs. Is it possible to classify all Cayley graphs which are QSRGs?

Bibliography

[1] M. Afkhami, H. R. Barani, K. Khashyarmanesh, and F. Rahbarnia. A new class of cayley graphs. *J. Algebra Appl.*, 15(04):1650076, 2016.

[2] B. Alspach, D. Marušič, and L. Nowitz. Constructing graphs which are 1/2-transitive. *J. Aust. Math. Soc.*, 56(3):391–402, 1994.

[3] B. Andrásfal. Graphentheoretische extremalprobleme. *Acta Math. Hung.*, 15:413–438, 1964.

[4] I. Bouwer. Vertex and edge transitive, but not 1-transitive, graphs. *Canad. Math. Bull.*, 13(2):231–237, 1970.

[5] J. Chen, C. H. Li, and Á. Seress. A family of half-transitive graphs. *Electron. J. Comb.*, 20(1):P56, 2013.

[6] H. Cheng and L. Cui. Tetravalent half-arc-transitive graphs of order p^5. *Appl. Math. Comput.*, 332:506–518, 2018.

[7] A. Deza and M. Deza. The ridge graph of the metric polytope and some relatives. In *Polytopes: Abstract, Convex and Computational*, pages 359–372. Springer, 1994.

[8] M. Erickson, S. Fernando, W. Haemers, D. Hardy, and J. Hemmeter. Deza graphs: A generalization of strongly regular graph. *J. Combin. Des.*, 7(6):395–405, 1999.

[9] Y.-Q. Feng, J. H. Kwak, X. Wang, and J.-X. Zhou. Tetravalent half-arc-transitive graphs of order $2pq$. *J. Algebraic Combin.*, 33:543–553, 2011.

[10] Y.-Q. Feng, J. H. Kwak, M.-Y. Xu, and J.-X. Zhou. Tetravalent half-arc-transitive graphs of order p^4. *European J. Combin.*, 29(3):555–567, 2008.

[11] C. Godsil and G. F. Royle. *Algebraic graph theory*, volume 207. Springer Science & Business Media, 2001.

[12] F. Goldberg. On quasi-strongly regular graphs. *Linear Multilinear Algebra*, 54(6):437–451, 2006.

[13] W. Golightly, W. Haynsworth, and D. Sarvate. A family of connected quasi-strongly regular graphs. *Congr. Numer.*, pages 89–96, 1997.

[14] D. F. Holt. A graph which is edge transitive but not arc transitive. *J. Graph Theory*, 5(2):201–204, 1981.

[15] F.-T. Hu, J.-W. Wang, and J.-M. Xu. A new class of transitive graphs. *Discrete Math.*, 310(4):877–886, 2010.

[16] D. Marušič. Hamiltonian circuits in cayley graphs. *Discrete Math.*, 46(1):49–54, 1983.

[17] S. B. Pejman, S. Payrovi, and A. Behtoei. Metric dimension of andrasfai graphs. *Opuscula Math.*, 39(3):415–423, 2019.

[18] W. Stein. Sage mathematics software. *http://www. sagemath. org/*, 2007.

[19] M. J. Stroppel. Generalizing the pappus and reye configurations. *Australas. J. Combin*, 72(2):249–272, 2018.

[20] W. T. Tutte. *Connectivity in graphs*. University of Toronto Press, 1966.

[21] X. Wang and Y.-Q. Feng. Hexavalent half-arc-transitive graphs of order $4p$. *European J. Combin.*, 30(5):1263–1270, 2009.

[22] J. Xie, D. Jia, and G. Zhang. On quasi-strongly regular graphs with parameters $(n, k, a; k-1, c_2)$. *Discrete Math.*, 344(3):112237, 2021.

[23] J. Xie and G. Zhang. On quasi-strongly regular graphs with parameters $(n, k, a; k-2, c_2)$ for a$< k-5$. *Discrete Math.*, 346(8):113433, 2023.

[24] C. Zhou and Y.-Q. Feng. An infinite family of tetravalent half-arc-transitive graphs. *Discrete Math.*, 306(18):2205–2211, 2006.

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